Product form steady-state distribution for Stochastic Automata Networks

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Stochastic Automata Networks in CT and DT

- To describe Multidimensional Markov Chains (Queueing Network, Stochastic Petri nets, SPA)
- The main idea is to obtain a generalized tensor description of the transition probability matrix or the transition rate matrix
- N finite automata. One automaton is used to model one component.
- The state space is included into the Cartesian product of the state space of the automata.
- The links of the automata carry information:
  - rate (in CT) or probability (DT) : fixed or function
  - local or synchronization

SAN and Tensor

- For CT-SAN, transition rate matrix is given by:

$$Q = \bigoplus_{g}{}_{i=1}^{N} Q_l^i + \sum_s \bigotimes_{g}{}_{i=1}^{N} Q_s^{(i)} + D$$

where D is a diagonal matrix (for normalization), $\bigoplus_g$ and $\bigotimes_g$ are the generalized tensor sum and the generalized tensor product and $Q_l^{(i)}$ and $Q_s^{(i)}$ are matrices describing the local transitions and transitions due to synchronization s on automaton i.

- Also proved for many formalisms (PEPA, Petri nets).
- for models without functions, use $\otimes$ and $\oplus$ instead of $\otimes_g$ and $\oplus_g$.
- For DT-SAN, transition probability matrix is given by:

$$P = \bigotimes_{g}{}_{i=1}^{N} P_l^i + \sum_s \bigotimes_{g}{}_{i=1}^{N} P_s^{(i)} + D$$

Generalized Tensor Product and Sum

- Ordinary tensor product $C = A \otimes B$ is defined by assigning the element of C that is in the (i_2, j_2) position of block (i_1, j_1), the value $a_{i_1 j_1} b_{i_2 j_2}$. We shall write this as

$$c_{\{(i_1,j_1);(i_2,j_2)\}} = a_{i_1 j_1} b_{i_2 j_2}.$$

- Generalized Tensor Product: Matrices of functions whose arguments are the states of the other components (ie the index of the matrix).

$$c_{\{(i_1,j_1);(i_2,j_2)\}} = a_{i_1 j_1}(i_2) b_{i_2 j_2}(i_1),$$

- As usual the sum is defined using the product:

$$D = A(\mathcal{B}) \oplus_g B(\mathcal{A}) \Leftrightarrow D = A(\mathcal{B}) \otimes_g Id_B + Id_A \otimes_g B(\mathcal{A}),$$

Compatibility between $\otimes$ (or $\otimes_g$) and $\times$

- associativity and distributivity: OK for $\otimes$ and $\otimes_g$.
- Most important property: Compatibility between ordinary product and ordinary tensor product.

$$(A \otimes B) \times (C \otimes D) = (A \times C) \otimes (B \times D)$$

- Imagine that A is not a matrix but a vector v ...
- What if v is in the kernel (null space) of C... or v is an eigenvector of C ?
- But the property is not valid for $\otimes_g$ in general.
- $0 \otimes v = 0$.

Product Form

- The steady-state distribution is the product of marginal distributions.

$$\pi(x_1, x_2, \ldots, x_n) = C\pi_{x_1}\pi_{x_2}\ldots\pi_{x_n}$$

- Proved for some queueing networks (Jackson, BCMP, G-networks), Petri-Nets, SAN, PEPA models
- First remark : a product of marginal distrbutions is a tensor product of vectors.

Motivation

- Links between (generalized or ordinary) tensor decomposition of multidimensional CTMC and product form for steady-state distribution.
- Is it possible for $M = \sum_i \otimes_{g\,j} P_j^i$ to have a steady-state solution $\pi = \otimes_j \pi_j$ where π_j are small vectors associated with P_j^i ?
- $$(\otimes_j \pi_j) \times \left(\sum_i \otimes_{\substack{g \\ j}} P_j^i \right) = 0$$
- Infinite State Space.

A little bit of history

- CT-Models with functions but without synchronization
  $Q = \bigoplus_{g\,i=1}^{N} Q_l^i$
  Generalization of many results (Hillston, Boucherie, Robertazzi) for Stochastic Petri Nets, Markov Chain in Competition, Modulated queues (ValueTools07, Performance Evaluation)
- Much harder in Discrete Time but still possible (Qest08)
- With synchronizations but without functions. First, give the number of components involved in an arbitrary synchronization:
  - 2: Master/Slave (like in a Jackson queueing network (PAPM 97), here with a new proof)
  - 3: Domino, like a G-network with triggered customer movements (Epew 2008).
- Properties of the generalized tensor product (Model 35)

Part I : Continous-Time

Master/slave synchronizations, no functions

$$Q = \bigoplus_{i=1}^{N} Q_l^i + \sum_s \bigotimes_{i=1}^{N} Q_s^{(i)} + D$$

Master-Slave: Matrix Description

- The master initiates the synchronization.
- The slave follows.
- A synchronization is described by 2 matrices.
  - Master description: $M^{(r)}$, a transition rate matrix.
  - Slave description: $E^{(r)}$, a transition probability matrix
- Local Transitions: transition rate matrices F_l.

Normalization

- All matrices are normalized, i.e. for all k we have:

$$M^{(r)}[k,i] \geq 0 \; if \; i \neq k \;\; and \quad \sum_i M^{(r)}[k,i] = 0,$$
$$E^{(r)}[k,i] \geq 0 \;\; and \quad \sum_i E^{(r)}[k,i] = 1,$$

- Normalization of the SAN: based on the normalization of the synchronizations (N^r) as the local transitions are already associated to transition rate matrices.
- Definition: Let M be a matrix, $diag(M)$ is a diagonal matrix whose elements are the diagonal elements of M.

A normalized tensor representation of a Master-Slave

1. $(M^{(r)} - diag(M^{(r)})) \otimes E^{(r)} \otimes I_1$: the slave accepts the synchronization.
2. $diag(M^{(r)}) \otimes I \otimes I_1$: normalization of term 1.

Main Result

Theorem: Consider a SAN with n automata and s with Master/Slave synchronizations. Consider matrices $\overline{M^{(r)}} = M^{(r)} - diag(M^{(r)})$ and $E^{(r)}$ associated to the description of synchronization r. Let g_l an eigenvector of $\overline{M^{(r)}}$. We assume that $g_l > 0$. Let Γ_r be the eigenvalue for matrix $\overline{M^{(r)}}$ associated to g_l.

If g_l is in the kernel of matrix $A_l = F_l + \sum_{r=1}^{R} \left(M^{(r)} 1_{msr(r)=l} + \Gamma_r (E^{(r)} - I) 1_{sl(r)=l} \right)$, then the steady-state distribution has a product form solution:

$$Pr(X_1, X_2, \cdots, X_n) = C \prod_{l=1}^{n} g_l(X_l), \qquad (1)$$

and C is a normalization constant.

Proof by tensor algebra

$(\otimes_l g_l) \times (\oplus_l F_l + \sum_i \otimes_l B_l^i)$	Reorganize the product (distributivity)
$\sum_i (\otimes_l g_l) \times (\otimes_l M_l^i)$	Then use compatibility with $\times$
$\sum_i (\otimes_l (g_l \times M_l^i))$	Assumptions on eigenvectors
	Factorize an find some matrices P_l^i
$\sum_i (\otimes_l (g_l \times P_l^i))$	such that
$\pi_{j0} P_{j0}^i = 0$	for all i, it exists $j0$ (automaton index)

Proof-2

The description of $(\pi_1 \otimes \pi_2 \otimes \ldots \otimes \pi_n)Q$ consists in 4 terms (two coming from the tensor sum, one for the Master/Slave and one for the normalization of the Master/Slave):

$$
\begin{aligned}
& (g_1 F_1 \otimes g_2 \otimes \ldots \otimes g_n) \\
+ & (g_1 \otimes g_2 F_2 \otimes \ldots \otimes g_n) \\
+ & (g_1 (M^{(r)} - diag(M^{(r)})) \otimes g_2 E^{(r)} \otimes \ldots \otimes g_n) \\
+ & (g_1 diag(M^{(r)}) \otimes g_2 I \otimes \ldots \otimes g_n)
\end{aligned}
$$

Proof-3

- Now remember that $g_1(M^{(r)} - diag(M^{(r)})) = g_1 \Gamma_r$. And of course $g_2 I = g_2$. After simplification, we get:

$$
\begin{aligned}
& (g_1 F_1 \otimes g_2 \otimes \ldots \otimes g_n) \\
+ & (g_1 \otimes g_2 F_2 \otimes \ldots \otimes g_n) \\
+ & (g_1 \Gamma_r \otimes g_2 E^{(r)} \otimes \ldots \otimes g_n) \\
+ & (g_1 diag(M^{(r)}) \otimes g_2 \otimes \ldots \otimes g_n)
\end{aligned}
$$

Proof-4

- Now, remark that $g_1\Gamma_r \otimes g_2 E^{(r)} = g_1 \otimes g_2 \Gamma_r E^{(r)}$ because the ordinary product is compatible with the tensor product.
- We factorize the first and the last terms and we do the same for the second and the third term. Furthermore we add and subtract the following term: $(g_1(M^{(r)} - diag(M^{(r)})) \otimes g_2 \otimes \ldots \otimes g_n)$.

$$\begin{array}{ll} & (g_1(F_1 + diag(M^{(r)})) \otimes g_2 \otimes \ldots \otimes g_n) \\ + & (g_1 \otimes g_2(F_2 + \Gamma_r E^{(r)}) \otimes \ldots \otimes g_n) \\ - & (g_1(M^{(r)} - diag(M^{(r)})) \otimes g_2 \otimes \ldots \otimes g_n) \\ + & (g_1(M^{(r)} - diag(M^{(r)})) \otimes g_2 \otimes \ldots \otimes g_n) \end{array}$$

Proof-5

- We factorize the first and the last term and we note that $g_1(M^{(r)} - diag(M^{(r)})) = g_1\Gamma_r$ to simplify the third term:

$$\begin{array}{ll} & (g_1(F_1 + M^{(r)}) \otimes g_2 \otimes g_3 \otimes \ldots \otimes g_n) \\ + & (g_1 \otimes g_2(F_2 + \Gamma_r E^{(r)}) \otimes \ldots \otimes g_n) \\ - & (g_1\Gamma_r \otimes g_2 \otimes \ldots \otimes g_n) \end{array}$$

- Again we use the compatibility of the ordinary product with the tensor product and we get after factorization:

$$\begin{array}{ll} & (g_1(F_1 + M^{(r)}) \otimes g_2 \otimes g_3 \otimes \ldots \otimes g_n) \\ + & (g_1 \otimes g_2(F_2 + \Gamma_r(E^{(r)} - I)) \otimes \ldots \otimes g_n) \end{array}$$

- This is the decomposition we need.

Example-1: G-network with + and - customers

- Infinite state space.
- Each automaton models the number of positive customers in a queue.
- The local transitions are the external arrivals (rate λ_l) and the departures to the outside (rate μ_l multiplied by probability d_l).
- Synchronization : departure of a customer on the master (the end of service with rate μ_l and probability $(1 - d_l)$), the departure of a customer on the slave if there is any.

Example-1-Matrix

$$\begin{aligned} Q^{(l)} &= \lambda_l(U - I) + \mu_l d_l(L - I^0), \\ M^{(r)} &= \mu_l(1 - d_l)(L - I^0), \\ E^{(r)} &= L \end{aligned} \tag{2}$$

- Tridiagonal matrix.
- π_l has a geometric distribution with rate ρ_l:

$$\rho_l = \frac{\lambda_l}{\mu_l + \sum_{r=1}^{R} \Gamma_r 1_{sl(r)=l}}.$$

- π_l is an eigenvector of operators $\overline{M^{(r)}}$ and $E^{(r)}$.
- Finally: $\Omega_r = \rho_{sl(r)}$ and $\Gamma_r = \rho_{msr(r)}\mu_{msr(r)}(1 - d_{msr(r)})$,

Part II : Continous-Time

fonctional rates, no synchronizations

$$Q = \bigoplus_{g\ i=1}^{N} Q_l^i$$

Here

- Infinite State Space.
- No Synchronizations.
- Functions to model the interactions between components.
- An easy model to represent multidimendional Markov chains:
  - without synchronized transition: only one component change during a transition.
  - with transition rates which are functions of the other components.
- $Q^{(l)}[k_l, i](\vec{k}, \vec{k} + (l, i))$: transition rate matrix for automaton l. The state of the automaton jumps from k_l to i. Due to this local jump, the global state changes from $\vec{k}$ to $\vec{k} + (l, i)$. The rate may depend of the global state (i.e. functionnal rate).

Example

- Consider a SAN with two automata $\mathcal{A}_1$ and $\mathcal{A}_2$.
- Both have a very simple state space: $\{0, 1\}$
- The transitions in $\mathcal{A}_1$ have a fixed rate $l1$ for the transition from 0 to 1 and $l2$ for the transition from 1 to 0.
- Automaton $\mathcal{A}_2$ has two functional transitions: the rate from 0 to 1 has a functional rate $f0$ and the reverse transitions has functional rate $f1$. Both functions use the state of automaton $\mathcal{A}_1$ as an argument (denoted as $x1$).
- $f0(x1) = mb + m(1-b)1_{x1=0}$ and $f1(x1) = m1 + m21_{x1=0}$.

Example-SAN

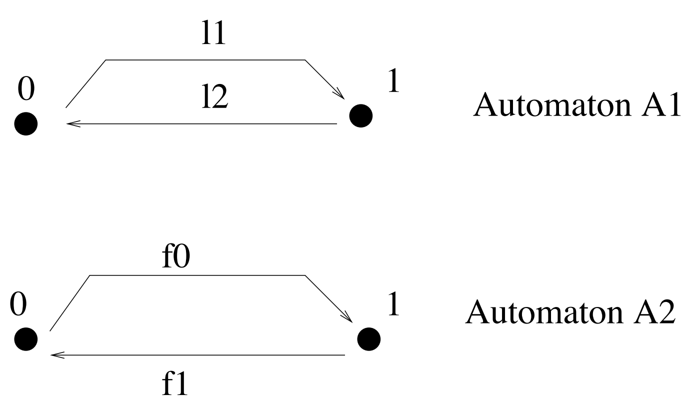


Figure 1: Stochastic Automata Network

Example

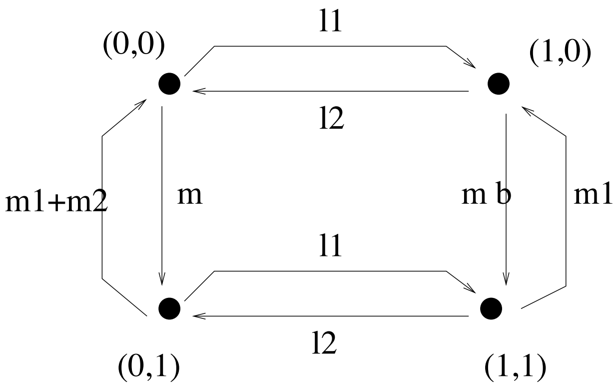


Figure 2: Markov chain

CK equation

$$Pr(\vec{k}) \quad \left[\sum_{l=1}^{n} \sum_{i \neq k_l} Q^{(l)}[k_l, i](\vec{k}, \vec{k} + (l,i)) \right] = \sum_{l=1}^{n} \sum_{i \neq k_l} Q^{(l)}[i, k_l](\vec{k} + (l,i), \vec{k}) Pr(\vec{k} + (l,i)). \tag{3}$$

Main Idea

- As the state space is discrete, functions can be replaced by an index.
- **Definition 1** *Let l be an automaton index, we consider all the functions in matrix $Q^{(l)}$ and we evaluate them for all state $\vec{k}$ when the transition from $\vec{k}$ to $\vec{k}+(l,i)$ takes place. Such a matrix will be denoted by $L^{(l,m(\vec{k}))}$ where $m(\vec{k})$ is an index. The set of matrices $L^{(l,m(\vec{k}))}$ will be denoted by $\mathcal{F}_{(l)}$.*

Definition

- **Definition 2** *Let α be a probability distribution. We note by $\mathcal{S}(\alpha)$ the set of transition rate matrices M such that $\alpha M = 0$ (i.e. α is in the the kernel of all matrices in $\mathcal{S}(\alpha)$).*
- **Property 1** *Interesting properties of $\mathcal{S}(\alpha)$:*
  1. **0** *(the matrix whose elements are all zero) is in $\mathcal{S}(\alpha)$*
  2. *$aM1$ is in $\mathcal{S}(\alpha)$. for all matrices $M1$ in $\mathcal{S}(\alpha)$ and a in R^+.*
  3. *$aM1 + bM2$ is in $\mathcal{S}(\alpha)$ for all matrices $M1$ and $M2$ in $\mathcal{S}(\alpha)$ and a, b in R^+ such that $a + b = 1$.*

Main Theorem

- **Theorem 1** *Consider a SAN with functions but without synchronizations. Assume that the steady state exists. If for each automaton l there exists a probability distribution π_l such that all the matrices in $\mathcal{F}_{(l)}$ are in $\mathcal{S}(\pi_l)$, then the SAN has a product form steady state distribution such that:*

$$Pr(x_0, .., x_n) = C\pi_1(x_1) \dots \pi_l x_l \pi_n(x_n).$$

- The proof is based on the resolution of the Chapman-Kolmogorov equation at steady-state.

$$Pr(\vec{k}) \quad \left[\sum_{l=1}^{n} \sum_{i \neq k_l} L^{(l,m(\vec{k}))}[k_l, i]\right] = \sum_{l=1}^{n} \sum_{i \neq k_l} L^{(l,m(\vec{k}))}[i, k_l] Pr(\vec{k} + (l, i)) \quad . \tag{4}$$

Corollary 1 *Consider the previous example. Matrices $M0$ and $M1$ have the same kernel if $b = \frac{m1}{m1+m2}$. If this condition is satisfied, the steady-state distribution of the SAN has product form:*

$$\pi(x1, x2) = C \left(\frac{l1}{l2}\right)^{x1} \left(\frac{m}{m1 + m2}\right)^{x2}.$$

Irreducibility

- The CTMC must be irreducible.
- It is sufficient that the components are irreducible but it is not necessary.
- If some matrices in $\mathcal{S}(\alpha)$ are reducible we can still obtain an irreducible CTMC with product form.

Example with reducible matrices

- A network with two automata $\mathcal{A}_1$ and $\mathcal{A}_2$.
- $\mathcal{A}_2$ has a very simple state space: $\{0, 1\}$
- $\mathcal{A}_1$ has four states
- The transitions in $\mathcal{A}_2$ have a fixed rate $l2$ for the transition from 0 to 1 and $m2$ for the transition from 1 to 0. Automaton $\mathcal{A}_1$ contains four functional transitions governed by four functions $f1$, $f2$, $f3$ and $f4$.

Example - SAN

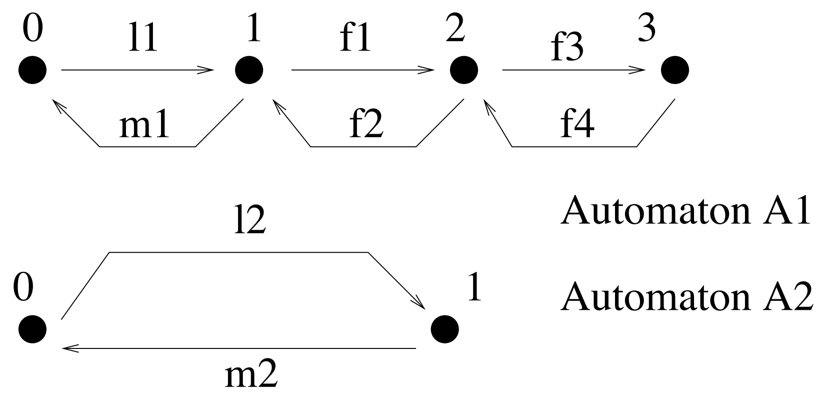


Figure 3: A more complex SAN with product form.

Example - Matrices

- $$\begin{cases} f1 = l1 \; 1_{x2=1} & f2 = m1 \; 1_{x2=1} \\ f3 = l1 \; 1_{x2=0} & f4 = m1 \; 1_{x2=0} \end{cases}$$

- $$M0 = \begin{pmatrix} -l1 & l1 & 0 & 0 \\ m1 & -m1 & 0 & 0 \\ 0 & 0 & -l1 & l1 \\ 0 & 0 & m1 & -m1 \end{pmatrix} \quad and \quad M1 = \begin{pmatrix} -l1 & l1 & 0 & 0 \\ m1 & -m1-l1 & l1 & 0 \\ 0 & m1 & -m1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Example - Kernels

- $M0 : \{u\ (m1, l1, 0, 0) + v\ (0, 0, m1, l1), \forall\ u, v \in \mathcal{R}\}$
- $M1 : \{u\ (1, l1/m1, l1^2/m1^2, 0) + v\ (0, 0, 0, 1), \forall\ u, v \in \mathcal{R}\}$.
- Clearly the vector $(1, l1/m1, l1^2/m1^2, l1^3/m1^3)$ is in both sets.
- Product form solution.

Tensor based proof

- The Hidden Lemma :
- **Property 2** *Let $A(\mathcal{B})$ and $B(\mathcal{A})$ be arbitrary functional transition rate matrices. Assume that w is in the kernel of $B(y)$ for every y and that w is positive. Similarly assume that there exists a positive vector v which is in the kernel of $A(x)$ for all x. Then we have:*

$$(v \otimes w) \times (A(\mathcal{B}) \oplus_g B(\mathcal{A})) = 0.$$

- Very simple proof (algebra).

Previous results

- Plateau's first theorem on product form for SAN
- Boucherie's first theorem on competing Markov chains.
- Verchere's theorem on modulated Markov Chains
- Partial Reversibility
- They are all corollaries of our main theorem.

Plateau's first theorem on Product Form SAN

- SAN with functions.
- The transition rate matrix of automaton l is the product of a function of $\vec{k}$ except component l ($f_l(\vec{k})$) by an usual transition rate matrix.
- $Q^{(l)}[i, k_l](\vec{k} + (l, i), \vec{k}) = f_l(\vec{k}) Q^{(l)}[i, k_l]$
- All these matrices have the same dominant eigenvector.

Boucherie's first theorem on competing MC

- Associated to Petri nets.
- A collection of Markov chains and a product process with restriction on the state space.
- Competition over ressources.
- Unformally: if a ressource is owned by component (i.e. a chain), transitions from some other chains (i.e. the competing ones) are removed.

Example

- Two chains $X1$ and $X2$ both with states $\{0, 1, 2, 3\}$ competing over one resource.
- Symmetrical rules.
- The resource is owned by a chain when it is in state 2 or 3.
- It is released when the chain jumps from state 3 to 1.
- Thus states in $\{2, 3\} \times \{2, 3\}$ are forbidden.
- When process $X1$ is in state 2 or 3 process $X2$ is stopped. If process $X1$ is in state 0 or 1, process $X2$ can move.

Graph of the example

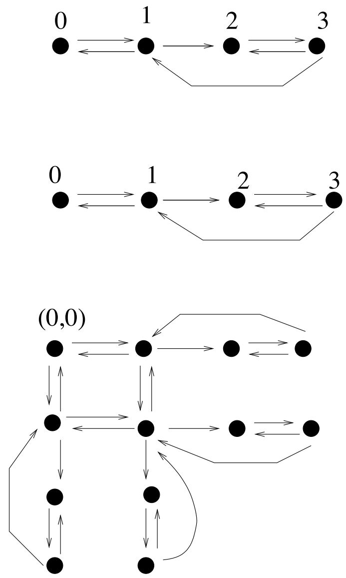


Figure 4: Two Markov chains in competition

Transitions of a competing Markov chain

- if states $\vec{k}$ and $\vec{k}'$ differ by more than 1 components, the transition rate is 0. (the transition matrix is a tensor sum of some matrices).
- from state $\vec{k}$ to state $\vec{k} + (l, i)$ the transition is the transition rate from k_l to i in chain l multiplied by an indicator function.
- This function is equal to zero when there exists a resource r owned by another chain which competes with l. (the transition rate matrices are the original matrices of the chains multiplied by a function of the states which takes value in $\{0, 1\}$.
- **This is a simple corollary of Plateau's first theorem where the functions take value in** $\{0, 1\}$**.**

Partial Reversibility

- **Definition 3** *An ergodic Markov chain W (matrix F) is partially reversible if and only if there exists a non empty subset X of the states such that for all states i and j in X we have local balance equations between i and j: $\pi(i)F(i,j) = \pi(j)F(j,i)$.*

- **Property 3** *Assume $F \in \mathcal{S}(\alpha)$, F irreducible and partially reversible with set X, then all matrices obtained from F after*

  1. *choosing any subset of X (let call it Y)*
  2. *multiplying all the transitions between states of Y by an arbitrary positive constant c.*

  are also in $\mathcal{S}(\alpha)$.

  We delete the transitions but we do not delete the states.

Partial Reversibility and Product Form

Theorem 2 *Consider a SAN with two automata $\mathcal{A}_1$ and $\mathcal{A}_2$. Assume that:*

1. *all the rates are constant in $\mathcal{A}_1$,*
2. *the matrix of $\mathcal{A}_2$ is partially reversible, (X: set of states of $\mathcal{A}_2$ with local balance, Y: an arbitrary subset of X, R: matrix of the transitions between states of Y),*
3. *$\mathcal{A}_2$ contains functions whose argument is the state of $\mathcal{A}_1$, and the functions are only carried by the transition between the states of Y.*
4. *R is the product of a function f by a constant matrix $R0$,*
5. *the CTMC is ergodic,*

then the SAN has a product form solution.

Checking Partial Reversibility

- Sometimes due to structural properties
- **Definition 4 (peninsula)** *Consider an ergodic CTMC associated to transition rate matrix F, a peninsula is a set of two nodes a, b such that:*
  - *Removing a and b disconnects the chain and creates two connected components A and B.*
  - $a \in A$ *and* $b \in B$.
  - b *is the only one successor of* a.
  - a *is the only one successor of* b.
- A peninsula implies a local balance between a and b.

Modulated network of queues

- One automata to represent the phase and one to represent the network of queues.
- Thus the synchronized transition between queues are local to the second automata.
- The transitions of the queues (not only the rate) may depend of the state of the phase.
- Verchère's theorem: if the steady-state distribution of the queueing network is always the same for all state of the phase, then the global system has a product form steady-state distribution.

Not that simple

- A two state phase.
- In phase 1, we have a Jackson network (transition (-1,+1)).
- In phase 2, a G-network with positive customers (transition (-1,+1)), triggers (transition (-1,-1,+1)) and negative customers (transition (-1,-1)).
- Both networks do not have the same transitions (because of negative customers and triggers).
- But if the rates are carefully chosen, they have the same geometric steady-state distribution
- Product-form.

Part III : Discrete-Time

fonctional rates, no synchronizations

$$P = \bigotimes_{g}{}_{i=1}^{N} P_l^i$$

Simple models for DT

- Infinite State Space.
- Local Events.
- Functions to model the interactions between components.
- Several components change during a transition (DT).
- With transition probabilities which are functions of the other components.
- Discrete-Time: $P = \bigotimes_{g}{}_{i=1}^{N} P_l^i$
  and P_l^i is a functional transition matrix
- remark that in Discrete-Time without function: $P = \bigotimes_{i=1}^{N} P_l^i$ has a product-form steady-state solution (independence)...

Functional Dependency Graph

- Functional Dependency Graph (FDG): directed graph (V, E)
- Node = Automaton.
- Directed edges $(A1, A2)$: automaton $A1$ uses the state of $A2$ in some functions to define rates or probabilities.
- The numerical algorithm developed by Plateau, Stewart, and Fernandes takes into account some properties of the Functional Dependency Graph.

Main Idea

- As the state space is discrete, functions can be replaced by an index.
- **Definition 5** *Let l be an automaton index, we consider all the functions in matrix $Q^{(l)}$ and we evaluate them for all state $\vec{k}$ when the transition from $\vec{k}$ to $\vec{k}+(l,i)$ takes place. Such a matrix will be denoted by $L^{(l,m(\vec{k}))}$ where $m(\vec{k})$ is an index. The set of matrices $L^{(l,m(\vec{k}))}$ will be denoted by $\mathcal{F}_{(l)}$.*

Definition

- **Definition 6** *Let α be a probability distribution. We note by $\mathcal{S}(\alpha)$ the set of transition rate matrices M such that $\alpha M = 0$ (i.e. α is in the the kernel of all matrices in $\mathcal{S}(\alpha)$).*
- The definition comes from CT models and it kept for compatibility reasons but it implies that we must transform transition probability matrix to use it.
- DT models: if two stochastic matrix $M1$ and $M2$ have the same dominant eigenvector α, then both $(M1 - Id)$ and $(M2 - Id)$ are in $\mathcal{S}(\alpha)$.
- **Property 4** *Interesting properties of $\mathcal{S}(\alpha)$:*
  1. **0** *(the matrix whose elements are all zero) is in $\mathcal{S}(\alpha)$*
  2. *$aM1 + bM2$ is in $\mathcal{S}(\alpha)$ for all matrices $M1$ and $M2$ in $\mathcal{S}(\alpha)$ and a, b in R^+ such that $a + b = 1$.*

Example

- Consider the functional matrix:

$$A(x) = \begin{bmatrix} 1 - x/2 & x/4 & x/4 \\ 0 & 1 - x/2 & x/2 \\ x & 0 & 1 - x \end{bmatrix}.$$

- If $0 < x < 1$, $A(x)$ is finite and irreducible.
- Steady-state distribution $(1/2, 1/4, 1/4)$ and it does not depend of the value of x.

Relations with the generalized tensor product

- **Property 5** *Let B be a positive matrix, let $A(\mathcal{B})$ be a matrix whose elements are functions of the index of B. Assume that w is an eigenvector of B with eigenvalue λ. Assume that for all states s of B, $A(s)$ has an eigenvector v associated to eigenvalue μ. Assume that both μ and v do not depend of s. Then we have:*

$$(v \otimes w) \times (A(\mathcal{B}) \otimes_g B) = \lambda\mu\,(v \otimes w)\,.$$

- Proof: simple algebra.
- Easy Generalization to an arbitrary number of automata....
- But Functional Dependency Graph = DAG...

Main result for Discrete Time

Theorem 3 *Consider a collection of functional stochastic matrices such that:*

- *The functional dependency graph is a Directed Acyclic Graph.*
- *For every matrix l there exists a positive vector π_l such that for every matrix index m, π_l is in the kernel of matrix $(Q^{(l,m)} - Id)$*
- *The Markov chain associated to the composition of these functional matrices is ergodic.*

Then the steady-state distribution has product form.

$$Pr(x_0, .., x_n) = C\pi_1(x_1)\ldots\pi_l(x_l)\pi_n(x_n).$$

What if the FDG is not a DAG

- Numerical Example.
- SAN with automata $\mathcal{A}_1$ and $\mathcal{A}_2$.
- States of $\mathcal{A}_1$ are labelled 1, 2 and 3.
- States of $\mathcal{A}_2$ are labelled 1 and 2.
- Transition matrix of $\mathcal{A}_1$ is functional matrix $A(x)$ where x is the state of $\mathcal{A}_2$ divided by 4.
- Transition matrix of $\mathcal{A}_2$ is functional matrix $D(y)$ where y is the state of $\mathcal{A}_1$ divided by 2.
- The dependency graph is a complete directed graph on two states and this is not a DAG.

Matrix $D(y)$

- $D(y) = \begin{bmatrix} 1-y/2 & y/2 \\ y/4 & 1-y/4 \end{bmatrix}$.
- If y is between 0 and 2, $D(y)$ is finite and irreducible.
- It has a steady-state distribution: $(1/3, 2/3)$ and it does not depend of the value of y.

Matrix of the SAN

- $$\begin{bmatrix} 0.65625 & 0.21875 & 0.046875 & 0.015625 & 0.046875 & 0.015625 \\ 0.09375 & 0.65625 & 0.015625 & 0.109375 & 0.015625 & 0.109375 \\ 0 & 0 & 0.4375 & 0.4375 & 0.0625 & 0.0625 \\ 0 & 0 & 0.1875 & 0.5625 & 0.0625 & 0.1875 \\ 0.0625 & 0.1875 & 0 & 0 & 0.1875 & 0.5625 \\ 0.1875 & 0.3125 & 0 & 0 & 0.1875 & 0.3125 \end{bmatrix}$$
- its steady-state distribution is $\pi = \begin{bmatrix} 0.1923 & 0.3169 & 0.0803 & 0.1664 & 0.0751 & 0.1690 \end{bmatrix}$,
- but the tensor product of the two steady-state distribution is:
$$\pi_A \otimes \pi_D = \begin{bmatrix} 1/6 & 1/3 & 1/12 & 1/6 & 1/12 & 1/6 \end{bmatrix}$$
- The SAN does not a have a product form distribution

What about $\oplus_g$

- **Property 6** *Let $A(\mathcal{B})$ and $B(\mathcal{A})$ be arbitrary functional transition rate matrices. Assume that w is in the kernel of $B(y)$ for every y and that w is positive. Similarly assume that there exists a positive vector v which is in the kernel of $A(x)$ for all x. Then we have:*

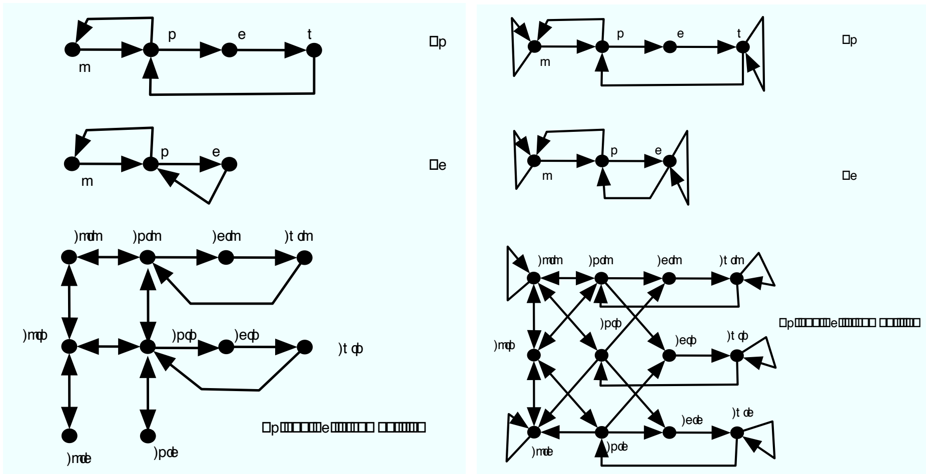
$$(v \otimes w) \times (A(\mathcal{B}) \oplus_g B(\mathcal{A})) = 0.$$

- Thus the result for CT-SAN is simpler (no restriction on the FDG)

Competing Markov chains in Discrete Time: Simple

- Chain $X1$ and $X2$ compete over one resource,
- if $X1$ has the resource, all the transitions of chain 2 are cancelled except self loops.
- if $X1$ does not own the resource, $X2$ evolves independently.
- $X2$ cannot block $X1$.
- When $X2$ owns the resource, $X1$ can move and if it takes the resource $X2$ is now blocked.

Competition in CT/DT



Differences between CT and DT

- CT models of competition have been proved to have product form (Boucherie 94).
- Strict priority (with preemption to take the resource) in DT.
- Race in CT.
- Priority implies (Functional Dependency graph = DAG)
- Several movements in DT, only one in CT.
- Cancellation of states in CT, open question in DT.
- Product form in both cases.

A more complex model of Competition/Collaboration

- H1': The resources are distributed among the X_i at every time slots.
- H2': If X_i receives b resources, it performs b steps of the Markov chain
- H3': Priority based allocation. X_1 has the highest priority. X_1 receives one resource.
- H4': According to the states of $X_1..X_{k-1}$, X_k receives b resources, the transition matrix of X_k is $(M_k)^b$
- H5': When no resources are given to X_k, it is blocked. Its transition matrix is Id.

Theorem 4 *Consider a collection of N chains $X_1, \ldots, X_N$ in competition over a set of R equivalent resources. Suppose that assumptions H1' to H5' are satisfied. Assume that the Markov chain of the DTMC modeling the competition is ergodic, then the steady-state distribution has product form.*

Part IV : Continous-Time

Domino Synchronizations, no functions

$$Q = \bigoplus_{i=1}^{N} Q_l^i + \sum_s \bigotimes_{i=1}^{N} Q_s^{(i)} + D$$

Domino Synchronization

- An **Ordered List of Automaton**.
- Domino effect: when a domino tile falls down, the next tile does the same (success)
- But the tile may stay up and the automata in the remaining part of the list does not move.

Domino Synchronization

Figure 5: Left: success: all the tiles are down; Right: the synchronization fails: the green tile is still up at the end.

Some definitions

- Here we only consider a domino synchronization with 3 automata.
- The automata are called the master, the slave and the relay.
- The master initiates the synchronization.
- The slave obeys and the synchronizations propagates to the relay, or the slave refuses and the relay does not move (but the master has changed its state).
- Finally the relay obeys (or maybe not).

Matrix Description

- A domino synchronization is described by 3 matrices.
  - Master description: $M^{(r)}$, a transition rate matrix.
  - Slave description: $E^{(r)}$, a transition probability matrix such that
  - Either $E^{(r)}[k,k] = 0$ (the synchro succeeds at this step) or $E^{(r)}[k,k] = 1$ (the synchro fails at this step).
  - If $E^{(r)}[k,k] = 0$ row k of $E^{(r)}$ gives the transition probability out of state k for the slave.
  - Relay description: T^r, a transition probability matrix (same assumptions as $E^{(r)}$).
  - $E^{(r)}$ is decomposed into $E_1^{(r)}$ (the slave obeys) and $E_2^{(r)}$ (the slave refuses)
- Local Transitions: transition rate matrices.

Normalization

- All matrices are normalized, i.e. for all k we have:

$$M^{(r)}[k,i] \geq 0 \; if \;\; i \neq k \;\; and \quad \sum_i M^{(r)}[k,i] = 0,$$
$$E^{(r)}[k,i] \geq 0 \;\; and \quad \sum_i E^{(r)}[k,i] = 1,$$
$$T^r[k,i] \geq 0 \;\; and \quad \sum_i T^r[k,i] = 1.$$

- Normalization of the SAN: based on the normalization of the synchronizations (N^r) as the local transitions are already associated to transition rate matrices.
- Definition: Let M be a matrix, $\sigma(M)$ is a diagonal matrix with the size of M such that for all index i, $\sigma(M)[i,i] = \sum_j M[i,j]$. As usual $diag(M)$ is a diagonal matrix whose elements are the diagonal elements of M.

A normalized tensor representation of a Domino

1. $(M^{(r)} - diag(M^{(r)})) \otimes E_1^{(r)} \otimes T^r \otimes I_1$: the slave accepts the synchronization.
2. $(M^{(r)} - diag(M^{(r)})) \otimes E_2^{(r)} \otimes I \otimes I_1$: the slave does not accept the synchronization.
3. $diag(M^{(r)}) \otimes \sigma(E_1^{(r)}) \otimes \sigma(T^r) \otimes I_1$: normalization of term 1.
4. $diag(M^{(r)}) \otimes \sigma(E_2^{(r)}) \otimes I \otimes I_1$: normalization of term 2.

Main Result

Theorem: Consider a SAN with n automata and s with Domino synchronizations. Consider matrices $\overline{M^{(r)}} = M^{(r)} - diag(M^{(r)})$ and $E^{(r)}$ associated to the description of synchronization r. Let g_l an eigenvector of both $E^{(r)}$ and $\overline{M^{(r)}}$. We assume that g_l exists. Let Ω_r (resp. Γ_r) be the eigenvalue for matrix $E^{(r)}$ (resp. $\overline{M^{(r)}}$) associated to g_l. If g_l is in the kernel of matrix

$A_l = F_l +$
$$\sum_{r=1}^{R} \left(M^{(r)} 1_{msr(r)=l} + \Gamma_r (E^{(r)} - \sigma(E^{(r)})) 1_{sl(r)=l} + \Gamma_r \Omega_r (T^r - \sigma(T^r)) 1_{rl(r)=l} \right);$$

then the steady-state distribution has a product form solution:

$$Pr(X_1, X_2, \cdots, X_n) = C \prod_{l=1}^{n} g_l(X_l), \tag{5}$$

and C is a normalization constant.

Sketch of the Proof-1

Main Idea: ALGEBRA...

$(\otimes_l g_l) \times (\oplus_l F_l + \sum_i \otimes_l B_l^i)$	Reorganize the product (distributivity)
$\sum_i (\otimes_l g_l) \times (\otimes_l M_l^i)$	Then use compatibility with $\times$
$\sum_i (\otimes_l (g_l \times M_l^i))$	Assumptions on eigenvectors
	Factorize an find some matrices P_l^i
$\sum_i (\otimes_l (g_l \times P_l^i))$	such that
$\pi_{j0} P_{j0}^i = 0$	for all i, it exists $j0$ (automaton index)

Sketch of the Proof-2

The description of $(\pi_1 \otimes \pi_2 \otimes \ldots \otimes \pi_n)Q$ consists in 7 terms (three coming from the tensor sum, two for the Domino and two for the normalization of the Domino):

$$
\begin{aligned}
& (\pi_1 F_1 \otimes \pi_2 \otimes \pi_3 \otimes \ldots \otimes \pi_n) \\
+ \quad & (\pi_1 \otimes \pi_2 F_2 \otimes \pi_3 \otimes \ldots \otimes \pi_n) \\
+ \quad & (\pi_1 \otimes \pi_2 \otimes \pi_3 F_3 \otimes \ldots \otimes \pi_n) \\
+ \quad & (\pi_1 (M^{(r)} - diag(M^{(r)})) \otimes \pi_2 E_1^{(r)} \otimes \pi_3 T^r \otimes \ldots \otimes \pi_n) \\
+ \quad & (\pi_1 (M^{(r)} - diag(M^{(r)})) \otimes \pi_2 E_2^{(r)} \otimes \pi_3 \otimes \ldots \otimes \pi_n) \\
+ \quad & (\pi_1 diag(M^{(r)}) \otimes \pi_2\, \sigma(E_1^{(r)}) \otimes \pi_3\, \sigma(T^r) \otimes \ldots \otimes \pi_n) \\
+ \quad & (\pi_1 diag(M^{(r)}) \otimes \pi_2\, \sigma(E_2^{(r)}) \otimes \pi_3 \otimes \ldots \otimes \pi_n)
\end{aligned}
$$

Sketch of the Proof-3

- $\pi_1(M^{(r)} - diag(M^{(r)})) = \pi_1\Gamma_r$.
- $\sigma(T^r) = I$
- $\sigma(E_1^{(r)}) + \sigma(E_2^{(r)}) = I$
- After simplification:

$$
\begin{array}{ll}
 & (\pi_1 F_1 \otimes \pi_2 \otimes \pi_3 \otimes \ldots \otimes \pi_n) \\
+ & (\pi_1 \otimes \pi_2 F_2 \otimes \pi_3 \otimes \ldots \otimes \pi_n) \\
+ & (\pi_1 \otimes \pi_2 \otimes \pi_3 F_3 \otimes \ldots \otimes \pi_n) \\
+ & (\pi_1 \Gamma_r \otimes \pi_2 E_1^{(r)} \otimes \pi_3 T^r \otimes \ldots \otimes \pi_n) \\
+ & (\pi_1 \Gamma_r \otimes \pi_2 E_2^{(r)} \otimes \pi_3 \otimes \ldots \otimes \pi_n) \\
+ & (\pi_1 diag(M^{(r)}) \otimes \pi_2 \otimes \pi_3 \otimes \ldots \otimes \pi_n)
\end{array}
$$

Sketch of the Proof-4

- Many algebraic manipulations (in the paper)...
- The ordinary product is compatible with the tensor product (i.e. $(\lambda A) \otimes B = A \otimes (\lambda B)$).
- $\sigma(E_2^{(r)}) = E_2^{(r)}$ and $\sigma(E_1^{(r)}) + \sigma(E_2^{(r)}) = I$. Using the distributivity, after cancellation we get:

$$
\begin{array}{ll}
 & (\pi_1 (F_1 + M^{(r)}) \otimes \pi_2 \otimes \pi_3 \otimes \ldots \otimes \pi_n) \\
+ & (\pi_1 \otimes \pi_2 (F_2 - \Gamma_r\, \sigma(E_1^{(r)})) \otimes \pi_3 \otimes \ldots \otimes \pi_n) \\
+ & (\pi_1 \otimes \pi_2 \otimes \pi_3 F_3 \otimes \ldots \otimes \pi_n) \\
+ & (\pi_1 \Gamma_r \otimes \pi_2 E_1^{(r)} \otimes \pi_3 T^r \otimes \ldots \otimes \pi_n)
\end{array}
$$

Proof

- We apply the assumption on the eigenvalue of $E_1^{(r)}$.
- After substitution:

$$\begin{array}{ll} & (\pi_1(F_1 + M^{(r)}) \otimes \pi_2 \otimes \pi_3 \otimes \ldots \otimes \pi_n) \\ + & (\pi_1 \otimes \pi_2(F_2 + \Gamma_r(E^{(r)} - \sigma(Er))) \otimes \pi_3 \otimes \ldots \otimes \pi_n) \\ + & (\pi_1 \otimes \pi_2 \otimes \pi_3(F_3 + \Gamma_r \Omega_r(T^r - \sigma(T^r))) \otimes \ldots \otimes \pi_n) \end{array}$$

- Now proceed with the other synchronizations...
- We obtain matrix A_l
- As $\pi_l A_l = 0$, we get $(\pi_1 \otimes \ldots \otimes \pi_l \otimes \ldots \otimes \pi_n)(Id \otimes A_l \otimes Id) = 0$.
- And the proof is complete.

Example-1: G-network with trigger

- Infinite state space.
- Each automaton models the number of positive customers in a queue.
- The local transitions are the external arrivals (rate λ_l) and the departures to the outside (rate μ_l multiplied by probability d_l).
- Synchronization : departure of a customer on the master (the end of service with rate μ_l and probability $(1 - d_l)$), the departure of a customer on the slave (a customer movement, if there is any), the arrival of a customer on the relay (always accepted).

Example-1-Matrix

$$Q^{(l)} = \lambda_l(U - I) + \mu_l d_l(L - I^0),$$
$$M^{(r)} = \mu_l(1 - d_l)(L - I^0), \quad (6)$$
$$E^{(r)} = L \ \ and \ \ T^r = U.$$

- Tridiagonal matrix.
- π_l has a geometric distribution with rate ρ_l:

$$\rho_l = \frac{\lambda_l + \sum_{r=1}^{R} \Omega_r \Gamma_r 1_{rl(r)=l}}{\mu_l + \sum_{r=1}^{R} \Gamma_r 1_{sl(r)=l}}.$$

- Because of its geometric distribution, π_l is an eigenvector of operators $\overline{M^{(r)}}$ and $E^{(r)}$.
- Finally: $\Omega_r = \rho_{sl(r)}$ and $\Gamma_r = \rho_{msr(r)}\mu_{msr(r)}(1 - d_{msr(r)})$,

Example 2-Three Deletions

- The deletion in the relay only occurs if the deletion in the slave was successful.

$$Q^{(l)} = \lambda_l(U - I) + \mu_l d_l(L - I^0),$$
$$M^{(r)} = \mu_l(1 - d_l)(L - I^0), \quad (7)$$
$$E^{(r)} = T^r = L.$$

- Tridiagonal matrix.
- π_l has a geometric distribution with rate ρ_l:

$$\rho_l = \frac{\lambda_l}{\mu_l + + \sum_{r=1}^{R} \Omega_r \Gamma_r 1_{rl(r)=l} + \sum_{r=1}^{R} \Gamma_r 1_{sl(r)=l}}.$$

and $\Omega_r = \rho_{sl(r)}$ and $\Gamma_r = \rho_{msr(r)}\mu_{msr(r)}(1 - d_{msr(r)})$.

Conclusion

- New results for Product Form for steady state distribution of multicomponent CTMC and DTMC
- As usual, DT is harder than CT
- Tensor based proof.
- Generalization to SAN with functions and synchronizations.
- Many competition/collaboration rules with product form.
- Links between partial balance and definition of $\mathcal{S}(\alpha)$.
- Generalization to SAN with functions and synchronizations.